Scaling laws of creep rupture of fiber bundles

Ferenc Kun,[1,*] Raul Cruz Hidalgo,[2] Hans J. Herrmann,[2] and Károly F. Pál[3]
[1]*Department of Theoretical Physics, University of Debrecen, P.O. Box 5, H-4010 Debrecen, Hungary*
[2]*Institute for Computational Physics, University of Stuttgart, Pfaffenwaldring 27, 70569 Stuttgart, Germany*
[3]*Institute of Nuclear Research (ATOMKI), P.O. Box 51, H-4001 Debrecen, Hungary*
(Received 13 September 2002; published 23 June 2003)

We study the creep rupture of fiber composites in the framework of fiber bundle models. Two fiber bundle models are introduced based on different microscopic mechanisms responsible for the macroscopic creep behavior. Analytical and numerical calculations show that above a critical load the deformation of the creeping system monotonically increases in time resulting in global failure at a finite time t_f, while below the critical load the system suffers only partial failure and the deformation tends to a constant value giving rise to an infinite lifetime. It is found that approaching the critical load from below and above the creeping system is characterized by universal power laws when the fibers have long-range interaction. The lifetime of the composite above the critical point has a universal dependence on the system size.

DOI: 10.1103/PhysRevE.67.061802 PACS number(s): 46.35.+z, 46.50.+a, 62.20.Mk

I. INTRODUCTION

Under high steady stresses fiber composites may undergo time-dependent deformation resulting in failure called creep rupture, which limits their lifetime, and hence, has a high impact on the applicability of these materials in construction elements. Both natural fiber composites like wood [1–4] and various types of fiber reinforced composites [5–10] show creep rupture phenomena, which have attracted continuous theoretical and experimental interest over the past years. The underlying microscopic failure mechanism of creep rupture is very complex depending on several characteristics of the specific types of materials, and is far from being well understood. Theoretical studies encounter various challenges: on the one hand, applications of fiber composites require the development of analytical and numerical models, which are able to predict the damage histories of loaded composites in terms of the specific parameters of constituents. On the other hand, creep rupture, similarly to other rupture phenomena, presents a very interesting problem for statistical physics, it is still an open problem to embed creep rupture into the general framework of statistical physics and to understand the analogy between rupture phenomena and phase transitions.

Creep failure tests are usually performed under uniaxial tensile loading when the specimen is subjected either to a constant load σ_0 or to an increasing load (ramp loading) and the time evolution of the damage process is monitored by recording the strain ε of the specimen and the acoustic signals emitted by microscopic failure events [1–10]. In the present paper we study the creep rupture of fiber composites in the framework of fiber bundle models [11–25] focusing on general characteristics of creep rupture. Our main goal is to reveal the universal aspects of a creeping system, which do not depend on specific material properties. We will derive scaling laws that emerge when approaching the failure stress both from below and above; furthermore, the finite-size scaling of the time to failure is studied. Beyond the theoretical understanding, exploring universal features of creep helps evaluate experimental data and can serve as a guide to extract the relevant information invariant under the variation of specific material properties.

We consider only the case of global load sharing (GLS) for the redistribution of load following fiber failure [13,18–24]. For several types of composite materials, GLS provides an adequate approach [26–30], and it has the advantage that many of the important GLS results can be obtained in closed analytic forms. During creep at the critical point and above it, stress localization occurs, which can be captured by localized load sharing in fiber bundle models [13,17,21], but this is beyond the scope of our investigations. The present study is focused on analytic modeling in the framework of GLS. We also develop efficient simulation techniques for testing the analytic results, furthermore, to study finite-size system. Since similar studies with local load sharing require large-scale computer simulations and a large amount of data processing, it will be presented in a forthcoming publication.

In a broad class of continuously reinforced materials no significant stress enhancement occurs around failed fibers so that a consistent interpretation of the damage process can be obtained in the framework of GLS [26–30]. Our theoretical results proved to be in a good qualitative agreement with recent experiments on the creep rupture life of metal matrix composites reinforced with brittle fibers [26–30].

II. MODELS

In order to work out a theoretical description of creep failure we introduce two fiber bundle models improving the classical fiber bundle models [11,12], which have proven very successful in the study of fracture of disordered materials [13–20,23,24]. The two models differ in the microscopic mechanisms assumed to be responsible for the macroscopic creep behavior: (1) in the first approach the fibers themselves are viscoelastic and they break when their deformation exceeds a stochastically distributed threshold value (strain controlled breaking), (2) in the second model the fi-

*Electronic address: feri@dtp.atomki.hu

 ©2003 The American Physical Society

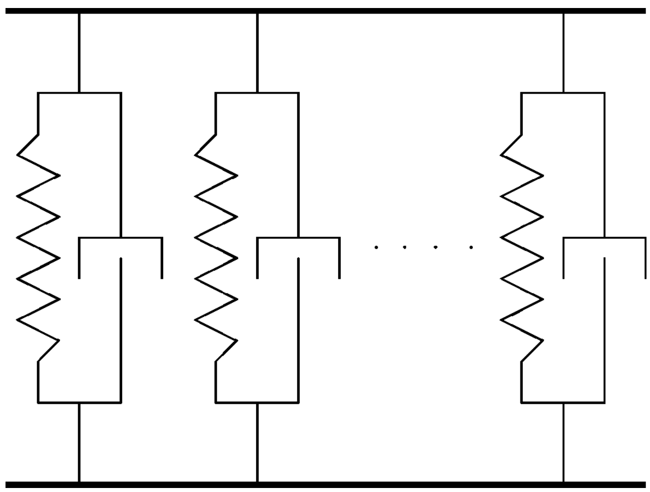

FIG. 1. The viscoelastic fiber bundle: intact fibers are modeled by Kelvin-Voigt elements. After fiber breaking the corresponding element is removed from the model.

bers are linearly elastic until they break stochastically in a stress-controlled way; however, after breaking, their relaxation is not instantaneous but the sliding of the broken fiber with respect to the matrix material, and the yielding or creeping of the matrix introduces an intrinsic time scale for the relaxation.

The mathematical formalism of the first model (viscoelastic bundle) is more simple; hence, the steps of analysis will be presented in details for this case. For the second model the corresponding results will be summarized.

A. Viscoelastic fiber bundle

1. Analytic model

Our model consists of N parallel fibers having viscoelastic constitutive behavior. For simplicity, the pure viscoelastic behavior of fibers is modeled by a Kelvin-Voigt element, which consists of a spring and a dashpot in parallel (see Fig. 1) and results in the constitutive equation

$$\sigma_o = \beta\dot{\varepsilon} + E\varepsilon, \tag{1}$$

where β denotes the damping coefficient and E is the Young modulus of fibers. Equation (1) provides the time-dependent deformation $\varepsilon(t)$ of a fiber at a fixed external load σ_o:

$$\varepsilon(t) = \frac{\sigma_o}{E}(1 - e^{-Et/\beta}) + \varepsilon_o e^{-Et/\beta}, \tag{2}$$

where ε_o denotes the initial strain at $t=0$. It can be seen that $\varepsilon(t)$ converges to σ_o/E for $t\to\infty$, which implies that the asymptotic strain fulfills Hook's law. To incorporate breaking in the model, we introduce a strain controlled failure criterion for fibers: a fiber fails during the time evolution of the system if its strain exceeds a damage threshold ε_d, which is an independent identically distributed random variable of fibers with probability density $p(\varepsilon_d)$ and cumulative distribution $P(\varepsilon_d) = \int_0^{\varepsilon_d} p(x)dx$. Due to the validity of Hook's law for the asymptotic strain values, the formulation of the failure criterion in terms of strain instead of stress implies that under a certain steady load the same amount of damage occurs as in the case of stress controlled failure; however, the breaking of fibers is not instantaneous but distributed over time. When a fiber fails its load has to be redistributed to the intact fibers. Assuming global load sharing, the time evolution of the system under a steady external load σ_o is finally described by the equation

$$\frac{\sigma_o}{1-P(\varepsilon)} = \beta\dot{\varepsilon} + E\varepsilon, \tag{3}$$

where the viscoelastic behavior of fibers is coupled to the failure of fibers in a global load sharing framework [31].

For the behavior of the solutions of Eq. (3) two distinct regimes can be distinguished depending on the value of the external load σ_o: When σ_o is below a critical value σ_c, Eq. (3) has a stationary solution ε_s, which can be obtained by setting $\dot{\varepsilon}=0$ in Eq. (3):

$$\sigma_o = E\varepsilon_s[1-P(\varepsilon_s)]. \tag{4}$$

It means that until this equation can be solved for ε_s at a given external load σ_o, the solution $\varepsilon(t)$ of Eq. (3) converges to ε_s when $t\to\infty$, and no macroscopic failure occurs. However, when σ_o exceeds the critical value σ_c no stationary solution exists; furthermore, $\dot{\varepsilon}$ remains always positive, which implies that for $\sigma > \sigma_c$ the strain of the system $\varepsilon(t)$ monotonically increases until the system fails globally at a time t_f [31].

In the regime $\sigma_o \leq \sigma_c$, Eq. (4) also provides the asymptotic constitutive behavior of the fiber bundle, which can be measured by controlling the external load σ_o and letting the system relax to ε_s. It follows from the above argument that the critical value of the load σ_c is the static fracture strength of the bundle, which can be determined from Eq. (4), as $\sigma_c = E\varepsilon_c[1-P(\varepsilon_c)]$, where ε_c is the solution of the equation $d\sigma_o/d\varepsilon_s|_{\varepsilon_c}=0$ [18]. Since $\sigma_o(\varepsilon_s)$ has a maximum value σ_c at ε_c, in the vicinity of ε_c it can be approximated as

$$\sigma_o \approx \sigma_c - A(\varepsilon_c - \varepsilon_s)^2, \tag{5}$$

where the multiplication factor A depends on the probability distribution P. A complete description of the system can be obtained by solving the differential equation (3). After separation of variables the integral arises

$$t = \beta \int d\varepsilon \frac{1-P(\varepsilon)}{\sigma_o - E\varepsilon[1-P(\varepsilon)]} + C, \tag{6}$$

where the integration constant C is determined by the initial condition $\varepsilon(t=0)=0$.

The creep rupture of the viscoelastic bundle can be interpreted so that for $\sigma_o \leq \sigma_c$ the system suffers only a partial failure that implies an infinite lifetime, $t_f=\infty$, and the emergence of a macroscopic stationary state, while above the critical load $\sigma_o > \sigma_c$ global failure occurs at a finite time t_f, which can be determined by evaluating the integral in Eq. (6) over the whole domain of definition of $P(\varepsilon)$.

Below the critical point $\sigma_o \leqslant \sigma_c$ the bundle relaxes to the stationary deformation ε_s through a decreasing breaking activity. To find the characteristic time scale of this relaxation process, the behavior of $\varepsilon(t)$ has to be analyzed in the vicinity of ε_s. It is useful to introduce a new variable δ as $\delta(t) = \varepsilon_s - \varepsilon(t)$. The governing differential equation of δ can be obtained from Eq. (3) by expanding it around ε_s:

$$\frac{d\delta}{dt} = -\frac{E}{\beta}\left[1 - \frac{\varepsilon_s p(\varepsilon_s)}{1-P(\varepsilon_s)}\right]\delta. \tag{7}$$

The solution of Eq. (7) has the form $\delta \sim \exp[-t/\tau]$, where τ is the characteristic time scale of the relaxation process:

$$\tau = \frac{\beta}{E}\frac{1}{\left[1 - \frac{\varepsilon_s p(\varepsilon_s)}{1-P(\varepsilon_s)}\right]}. \tag{8}$$

It is a very important question how the relaxation time τ changes when the external driving approaches the critical point σ_c from below. Based on Eq. (5) it can be simply shown that

$$\tau \sim (\sigma_c - \sigma_o)^{-1/2} \quad \text{for} \quad \sigma_o < \sigma_c, \tag{9}$$

which means that approaching the critical point from below the relaxation time of the system diverges according to a universal power law with an exponent $-1/2$ independent of the form of disorder distribution. Note that a similar power-law divergence of the number of successive relaxation steps was found in Refs. [32,33] for a dry fiber bundle subjected to a constant load.

Above the critical point the behavior of the lifetime of the system can be analyzed analogously when σ_o goes to σ_c from above. When σ_o is in the vicinity of σ_c, i.e., $\sigma_o = \sigma_c + \Delta\sigma_o$, where $\Delta\sigma_o \ll \sigma_c$, it can be expected that the curve of $\varepsilon(t)$ falls very close to ε_c for a very long time and the breaking of the system occurs suddenly. Hence, the total time to failure, i.e., the integral in Eq. (6), is dominated by the region close to ε_c when $\Delta\sigma_o$ is small. Making use of the power series expansion, Eq. (5), the integral in Eq. (6) can be rewritten as

$$t_f \approx \beta \int d\varepsilon \frac{1-P(\varepsilon)}{\Delta\sigma_o - A(\varepsilon_c - \varepsilon)^2}, \tag{10}$$

which has to be evaluated over a small ε interval in the vicinity of ε_c. After performing the integration it follows

$$t_f \approx (\sigma_o - \sigma_c)^{-1/2} \quad \text{for} \quad \sigma_o > \sigma_c. \tag{11}$$

Thus, t_f has a power-law divergence at σ_c with a universal exponent $-\frac{1}{2}$ independent of the specific form of the disorder distribution $P(\varepsilon)$, similar to τ below the critical point.

2. Finite size effect

In the above analytic treatment the size of the system, i.e., the number of fibers in the bundle, is infinite. However, it can be expected that the lifetime of a finite bundle has a nontrivial size scaling even in the case of global load sharing. During the creep rupture process the fibers of a finite bundle break one by one in the increasing order of their breaking thresholds. Let $\varepsilon_j, j=1,\ldots,N$ denote the breaking thresholds assigned to the fibers in a realization of the bundle. Since fibers break one by one, the actual load on the intact fibers after the failure of i fibers is $\sigma_i = \sigma_o N/(N-i)$, where $i=0,\ldots,N-1$, and the time $\Delta t(\varepsilon_i, \varepsilon_{i+1})$ between the breaking of the ith and $(i+1)$th fibers (in the ordered series) reads as

$$\Delta t(\varepsilon_i, \varepsilon_{i+1}) = -\frac{\beta}{E}\left[\ln\left(\varepsilon_{i+1} - \frac{\sigma_i}{E}\right) - \ln\left(\varepsilon_i - \frac{\sigma_i}{E}\right)\right]. \tag{12}$$

The lifetime t_f of a sample of N fibers takes the form

$$t_f(N) = \sum_{i=0}^{N-1} \Delta t(\varepsilon_i, \varepsilon_{i+1}). \tag{13}$$

In order to determine how t_f depends on N, Eq. (13) has to be averaged over many realizations of the disorder distribution, which can be performed analytically. The details of the analytic calculations are summarized in the Appendix. Finally, the average lifetime $\langle t_f(N)\rangle$ of a bundle of N fibers can be cast in the form

$$\langle t_f(N)\rangle \approx t_f(\infty) + \frac{\beta\sigma_o}{N}\int \frac{d\varepsilon E\varepsilon P(\varepsilon)[1-P(\varepsilon)]}{\{\sigma_o - E\varepsilon[1-P(\varepsilon)]\}^3}. \tag{14}$$

Equation (14) shows that for finite bundles the average lifetime $\langle t_f(N)\rangle$ converges to the lifetime of the infinite bundle $t_f(\infty)$ as $\sim 1/N$ with increasing number of fibers N. It is interesting to note that in the case of global load sharing the average strength of the bundle σ_c does not have a size dependence.

3. Simulation technique

Most of the analytic results of the preceding sections, except for the finite-size scaling of t_f, were obtained for infinite bundles. Computer simulations of the creep rupture of finite bundles are needed to justify the validity of analytic predictions for finite systems and to be able to model the rupture process of realistic finite systems.

In the framework of GLS an efficient simulation technique can be worked out for the failure process. Based on the arguments of the preceding subsection, the GLS simulation of the creep process of a bundle of N fibers proceeds as follows: (1) Random breaking thresholds ε_i, $i=1,\ldots,N$ are drawn from a probability distribution p, then the thresholds are put into increasing order. (2) The time $\Delta t(\varepsilon_i, \varepsilon_{i+1})$ between the breaking of the ith and $(i+1)$th fibers is calculated according to Eq. (12). (3) The time elapsed till the breaking of the ith fiber is obtained as $t(\varepsilon_i)$

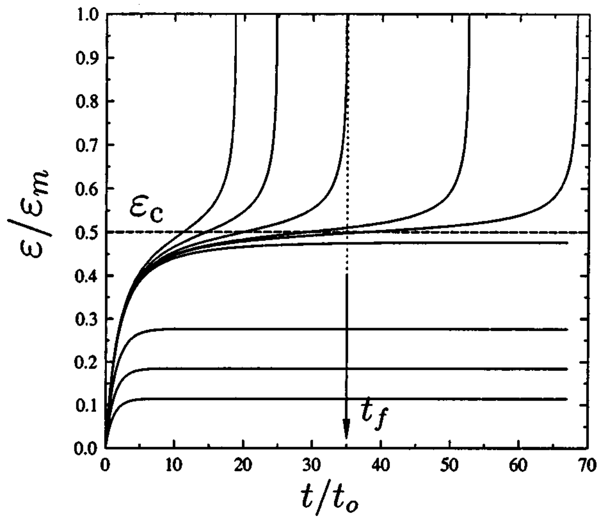

FIG. 2. $\varepsilon(t)$ for several values of σ_o below and above σ_c for a bundle of 10^7 fibers. The critical strain ε_c and the time to failure t_f for one example are indicated. t_o denotes the characteristic time $t_o=\beta/E$ of the system.

$=\Sigma_{j=0}^{i-1}\Delta t(\varepsilon_i,\varepsilon_{i+1})$, from which the deformation as a function of time $\varepsilon(t)$ can be determined by inversion. The lifetime t_f of the finite bundle can be obtained making use of Eq. (13).

For simulations we considered a uniform distribution of failure thresholds between 0 and a maximum value ε_m with the probability density function $p(\varepsilon)=1/\varepsilon_m$ and distribution function $P(\varepsilon)=\varepsilon/\varepsilon_m$. In this case the stationary solution, the critical load and the corresponding critical strain can be obtained as $\sigma_o=E\varepsilon(1-\varepsilon/\varepsilon_m)$, $\sigma_c=E\varepsilon_m/4$, $\varepsilon_c=\varepsilon_m/2$, respectively. $\varepsilon_m=1$ was set in all the simulations. The deformation-time diagram $\varepsilon(t)$ obtained by simulations is presented in Fig. 2 for several different values of σ_o below and above σ_c. The two regimes depending on the value of the external load σ_o can be clearly distinguished.

To test the validity of the universal power-law behavior of t_f as a function of the distance from the critical load given by Eq. (11), simulations were performed with various disorder distributions, i.e., besides the uniform distribution the Weibull distribution of the form $P(\varepsilon)=1-\exp[-(\varepsilon/\lambda)^\rho]$ was employed. The value of the characteristic strain λ was set to 1, and the shape of the distribution was controlled by varying the value of ρ. The results are presented in Fig. 3, where an excellent agreement of the simulations and the analytic results can be observed. Figure 3 supports that the exponent of t_f as a function of $\sigma_o-\sigma_c$ is universal, it does not depend on the specific form of the disorder distribution.

To study the finite-size scaling of the time to failure t_f, a uniform distribution was used for the failure thresholds. The value of the external load was fixed above σ_c and the number of fibers N was varied from 5×10^2 to 10^7. Averages were calculated over 10^4 samples for each system size N. The results obtained by simulations are presented in Fig. 4, where an excellent agreement of simulations and analytic results can be observed for four orders of magnitude in the system size N.

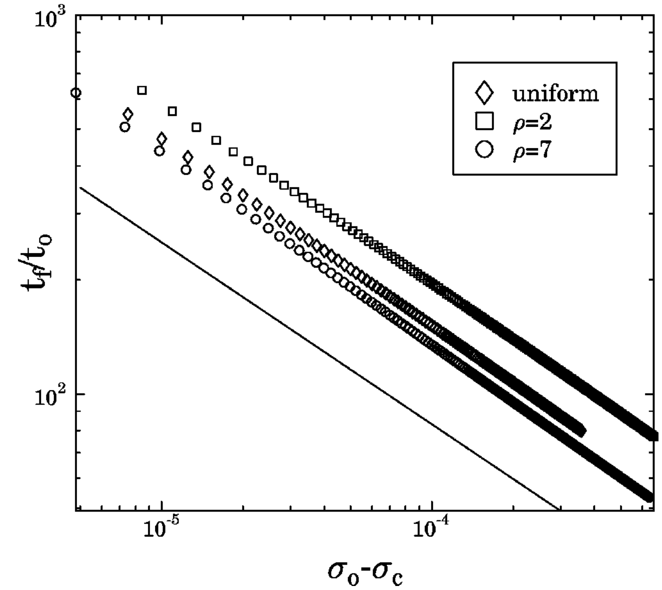

FIG. 3. The lifetime of a bundle of 10^7 fibers as a function of $\sigma_o-\sigma_c$ above the critical point for three different disorder distributions, i.e., uniform and Weibull distributions with $\rho=2,7$ have been considered. The straight line of slope -0.5 is drawn to guide the eye.

B. Slowly relaxing fibers

1. Analytic model

Another important microscopic mechanism that can lead to macroscopic creep is the slow relaxation following fiber failure. In this case, the components of the solid are linearly elastic until they break; however, after breaking, they undergo a slow relaxation process, which can be caused, for instance, by the sliding of broken fibers with respect to the

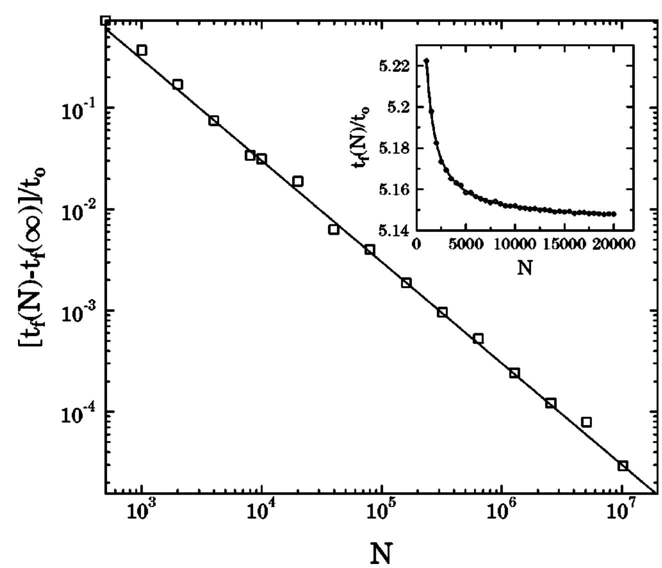

FIG. 4. The size-dependent lifetime of the bundle. The inset shows the lifetime of relatively small systems on a linear plot, while in the main figure the difference of the lifetime of finite bundles and the infinite one obtained from Eq. (14) can be seen on a double logarithmic plot. The slope of the fitted straight line is -0.99 ± 0.02.

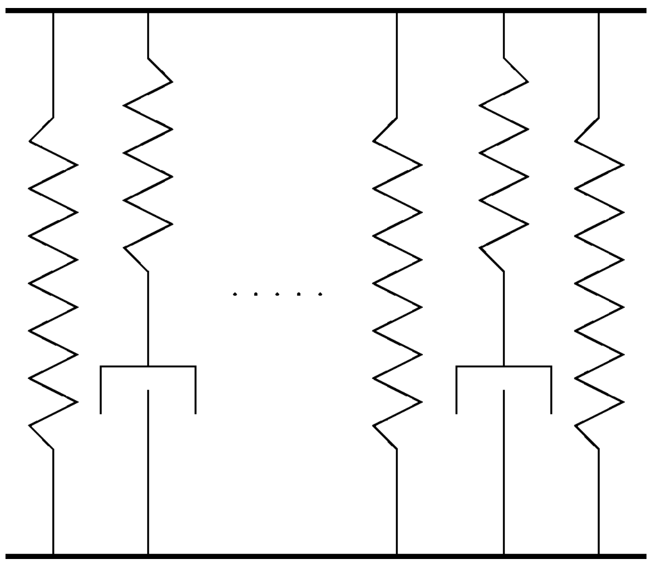

FIG. 5. The model solid when intact fibers are linearly elastic, and the broken ones with the surrrounding matrix are modeled by Maxwell elements.

matrix material or by the creeeping matrix. To take into account this effect, our approach is based on the model introduced in Refs. [9,10], where the response of a viscoelastic-plastic matrix reinforced with elastic and also viscoelastic fibers have been studied. The model consists of N parallel fibers, which break in a stress controlled way, i.e., subjecting a bundle to a constant external load fibers break during the time evolution of the system when the local load on them exceeds a stochastically distributed breaking threshold $\sigma_i, i=1,\ldots,N$. Intact fibers are assumed to be linearly elastic, i.e., $\sigma = E_\mathrm{f}\varepsilon_\mathrm{f}$ holds until they break, and hence, for the deformation rate it applies

$$\dot{\varepsilon}_\mathrm{f} = \frac{\dot{\sigma}}{E_\mathrm{f}}. \tag{15}$$

Here ε_f denotes the strain and E_f is the Young modulus of intact fibers. The main assumption of the model is that when a fiber breaks its load does not drop to zero instantaneously, instead it undergoes a slow relaxation process introducing a time scale into the system. In order to capture this effect, the broken fibers with the surrounding matrix material are modeled by Maxwell elements, as illustrated in Fig. 5, i.e., they are conceived as a serial coupling of a spring and a dashpot that results in a nonlinear response

$$\dot{\varepsilon}_\mathrm{b} = \frac{\dot{\sigma}_\mathrm{b}}{E_\mathrm{b}} + B\sigma_\mathrm{b}^\mathrm{m}, \tag{16}$$

where σ_b and ε_b denote the time-dependent load and deformation of a broken fiber, respectively. The relaxation of the broken fiber is characterized by three parameters E_b, B, and m, where E_b is the effective stiffness of a broken fiber, and the exponent m characterizes the strength of nonlinearity of the element. We study the behavior of the system for the region $m \geqslant 1$.

Assuming global load sharing for the load redistribution, the constitutive equation describing the macroscopic elastic behavior of the composite reads

$$\sigma_\mathrm{o} = \sigma(t)[1-P(\sigma(t))] + \sigma_\mathrm{b}(t)P(\sigma(t)). \tag{17}$$

Equation (17) takes into account that broken fibers carry also a certain amount of load $\sigma_b(t)$; furthermore, $P(\sigma(t))$ and $1-P(\sigma(t))$ denote the fraction of broken and intact fibers at time t, respectively [34]. It can be seen from Eq. (17) that under a constant external load σ_0, the load of intact fibers σ will also be time dependent due to the slow relaxation of the broken ones.

Due to the boundary condition illustrated in Fig. 5, the two time derivatives have to be always equal

$$\dot{\varepsilon}_\mathrm{f} = \dot{\varepsilon}_\mathrm{b}. \tag{18}$$

The differential equation governing the time evolution of the system can be obtained by expressing σ_b in terms of σ from Eq. (17) and substituting it into Eq. (16) and finally into Eq. (18),

$$\dot{\sigma}\left\{\frac{1}{E_\mathrm{f}} - \frac{1}{E_\mathrm{b}}\left[1 - \frac{1}{P(\sigma)} + \frac{p(\sigma)}{P(\sigma)^2}(\sigma - \sigma_\mathrm{o})\right]\right\} = B\left[\frac{\sigma_\mathrm{o} - \sigma[1-P(\sigma)]}{P(\sigma)}\right]^m. \tag{19}$$

In order to determine the initial condition for the integration of Eq. (19) the breaking process of fibers has to be analyzed. Subjecting the undamaged specimen to an external stress σ_o all the fibers attain this stress value immediatelly due to the linear elastic response. Hence, the time evolution of the system can be obtained by integrating Eq. (19) with the initial condition $\sigma(t=0) = \sigma_o$. Since intact fibers are linearly elastic, the deformation-time history $\varepsilon(t)$ of the model can be deduced as $\varepsilon(t) = \sigma(t)/E_f$, that has an initial jump to $\varepsilon_o = \sigma_o/E_f$. It follows that those fibers that have breaking thresholds σ_i smaller than the externally imposed σ_o immediately break.

To characterize the macroscopic behavior of the composite the solutions $\sigma(t)$ of Eq. (19) have to be analyzed at different values of the external load σ_o. Similar to the previous model, two different regimes of $\sigma(t)$ can be distinguished depending on the value of σ_o: if the external load falls below a critical value σ_c a stationary solution σ_s of the governing equation exists, which can be obtained by setting $\dot{\sigma}=0$ in Eq. (19),

$$\sigma_o = \sigma_s[1-P(\sigma_s)]. \tag{20}$$

This means that until Eq. (20) can be solved for σ_s the solution $\sigma(t)$ of Eq. (19) converges asymptotically to σ_s, resulting in an infinite lifetime t_f of the composite. Note that Eq. (20) provides also the asymptotic constitutive behavior of the model, which can be measured by quasistatic loading. If the external load falls above the critical value, the deformation rate $\dot{\varepsilon} = \dot{\sigma}/E_f$ remains always positive resulting in a macroscopic rupture in a finite time t_f. It follows from Eq. (20) that the critical load σ_c of creep rupture coincides with the static fracture strength of the composite.

The behavior of the system shows again universal aspects in the vicinity of the critical point. Below the critical point the relaxation of $\sigma(t)$ to the stationary solution σ_s is governed by a differential equation of the form

$$\frac{d\delta}{dt} \sim \delta^m, \tag{21}$$

where δ denotes the difference $\delta(t) = \sigma_s - \sigma(t)$. Hence, the characteristic time scale τ of the relaxation process only emerges if $m=1$; furthermore, in this case also $\tau \sim (\sigma_c - \sigma_o)^{-1/2}$ holds when approaching the critical point. However, for $m>1$ the relaxation process is characterized by $\delta(t) = at^{1/1-m}$, where $a \to 0$ with $\sigma_0 \to \sigma_c$.

Similar to the previous model, it can also be shown that the lifetime t_f of the bundle has a power-law divergence when the external load approaches the critical point from above,

$$t_f \sim (\sigma_0 - \sigma_c)^{-(m-1/2)} \quad \text{for} \quad \sigma_o > \sigma_c. \tag{22}$$

The exponent is universal in the sense that it is independent of the disorder distribution; however, it depends on the stress exponent *m*, which characterizes the nonlinearity of broken fibers.

2. Simulation technique

Subjecting a finite bundle of *N* fibers to σ_o external stress, those fibers whose failure threshold falls below σ_o break immediately. The number N_o of intially breaking fibers can be estimated from the disorder distribution as $N_o \approx NP(\sigma_o)$. In the presence of broken fibers the system slows down and the remaining fibers of the bundle break one by one in the increasing order of their breaking thresholds, $\sigma_{N_o+1} < \sigma_{N_o+2} < \sigma_N$. In order to construct an efficient simulation technique one has to determine the time elapsed between two consecutive breakings during the creep process.

The macroscopic constitutive equation for a system of *N* fibers when *i* fibers have already failed can be written as

$$\sigma_o = \sigma \frac{N-i}{N} + \sigma_b \frac{i}{N}. \tag{23}$$

Making use of Eqs. (16) and (18), the differential equation describing the time evolution of the load of intact fibers σ can be cast in the form

$$\dot{\sigma}\left[\frac{1}{E_f} - \frac{1}{E_b}\left(1 - \frac{N}{i}\right)\right] = B\left(\frac{N}{i}\right)^m f_i(\sigma)^m, \tag{24}$$

where $f_i(x)$ is introduced for brevity as

$$f_i(x) = \sigma_o - \frac{N-i}{N}x. \tag{25}$$

The time $\Delta t(\sigma_i, \sigma_{i+1})$ elapsed between the breaking of the ith and $(i+1)$th fibers can be determined by integrating Eq. (24) from σ_i to σ_{i+1}, which yields for $m \neq 1$,

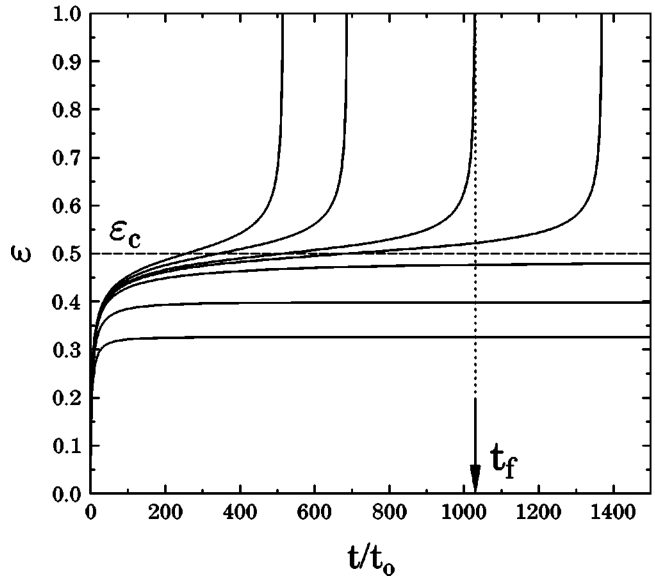

FIG. 6. ε as a function of *t* for several values of σ_o below and above σ_c. $N = 10^7$ fibers were used.

$$\Delta t(\sigma_i, \sigma_{i+1}) = \frac{K_i}{(m-1)}[f_i(\sigma_{i+1})^{1-m} - f_i(\sigma_i)^{1-m}], \tag{26}$$

and the multiplication factor K_i reads as

$$K_i = \frac{N}{N-i}\left(\frac{i}{N}\right)^m \frac{1}{B}\left[\frac{1}{E_f} - \frac{1}{E_b}\left(1 - \frac{N}{i}\right)\right]. \tag{27}$$

For $m=1$ the corresponding equation has the form

$$\Delta t(\sigma_i, \sigma_{i+1}) = K_i[\ln f_i(\sigma_{i+1}) - \ln f_i(\sigma_i)]. \tag{28}$$

Then the simulation proceeds as in the case of viscoelastic bundles, but in the above formulas the number of broken fibers i varies as $i = N_o, N_o+1, \ldots, N-1$, so the time as a function of σ can be obtained as $t(\sigma_i) = \sum_{j=N_o+1}^{i} \Delta t(\sigma_j, \sigma_{j+1})$ from which the deformation as a function of time $\varepsilon_i(t)$ can be determined, since $\sigma = E_f \varepsilon_f$ always holds. The lifetime t_f of the system can be obtained by summing up all the Δt's.

For the purpose of explicit calculations a uniform distribution was prescribed for the breaking thresholds σ_i between 0 and 1. The deformation as a function of time is plotted in Fig. 6 for several different values of the external load below and above the critical load. Similar to the previous model, the two regimes of the creeping system can be clearly distinguished.

To study the behavior of the time to failure as a function of the distance from the critical point, simulations were performed for several different values of the exponent *m*. In Fig. 7 the results are presented for $m=1.5$ and $m=2.5$. The slope of the fitted straight lines agrees very well with the analytic predictions of Eq. (22).

The size scaling of time to failure t_f was analyzed by simulating the creep rupture of bundles of size $N = 5\times10^2 - 10^7$, setting a uniform distribution for the breaking thresh-

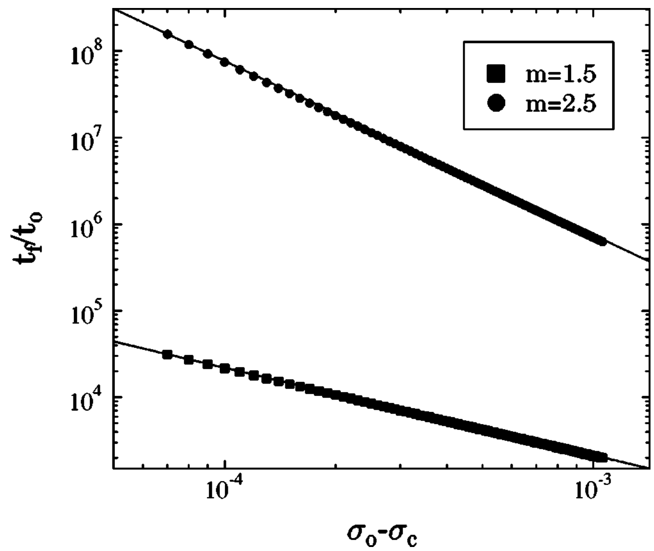

FIG. 7. Lifetime t_f as a function of the distance from the critical point $\sigma_o - \sigma_c$ for two different values of the parameter *m*. The number of fibers in the bundle was taken $N = 10^7$.

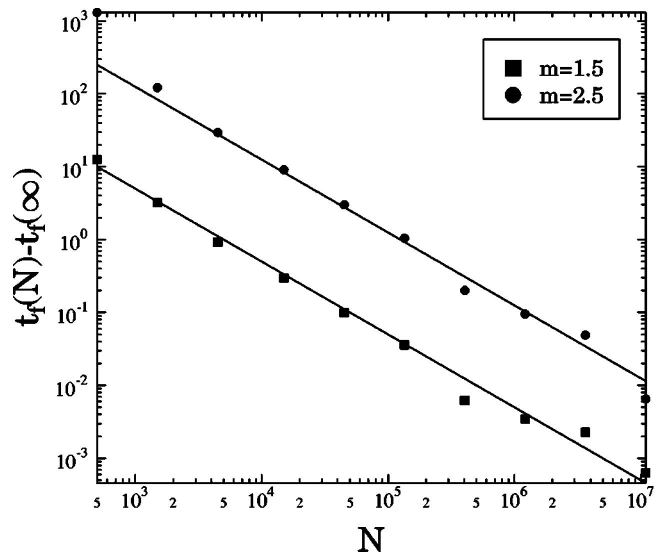

FIG. 8. Size dependence of the lifetime t_f for two different values of the parameter *m*.

olds. We found that $t_f(N)$ converges to the lifetime of the infinite system $t_f(\infty)$ according to the universal law, Eq. (14), independent of the value of the exponent *m*. In Fig. 8 the best fit was obtained for both curves with slope -1.0 ± 0.05 for both *m* values.

III. CONCLUSIONS

The creep rupture of fibrous materials occurring under a steady external load is microscopically a rather complex phenomenon depending on a diversity of possible material specific mechanisms. Therefore, on the one hand, it is impossible to work out a general theoretical framework that takes into account all the features of the process and has predictive power and, on the other hand, it is very important to reveal universal aspects of the creep process, which do not depend on specific material properties relevant at the microlevel.

In the present paper we studied the creep rupture of fibrous materials in the framework of fiber bundle models taking into account two possible microscopic mechanisms of creep: (1) In the first approach the fibers themselves are viscoelastic and they break when their deformation exceeds a stochastically distributed threshold value. (2) In the second model the fibers are linearly elastic until they break; however, after breaking, their relaxation is not instantaneous but the creeping matrix introduces an intrinsic time scale for the relaxation. The first model can be relevant for natural fiber composites such as wood, which are composed of viscoelastic fibers [1–4], while the second model can provide an adequate description of metal matrix composites reinforced by brittle fibers [26–30]. Analytical and numerical calculations showed in both models that increasing the external load in a specimen, a transition takes place from a partially failed state of infinite lifetime to a state where global failure occurs at a finite time. The critical load turned to be the static fracture strength of the material. It was found that irrespective of the details of the two model systems, universal behavior emerges when the creeping system approaches the critical state.

From practical point of view the knowledge of universal aspects of creep response of materials is very important for materials design, in order to predict creep life of structural elements and to design composites with higher lifetime. Recently, extensive experiments have been carried out to determine the dependence of lifetime of metal matrix composites reinforced with ceramic fibers on the value of the external load [26–30]. The results obtained are in good qualitative agreement with our theoretical predictions, i.e., a power-law behavior of t_f as a function of the distance from the critical load was revealed with an exponent that is independent of the distribution of fiber strength but depends on the stress exponent *m* of the matrix material [28,30].

ACKNOWLEDGMENTS

This work was supported by the project SFB381, by the NATO Grant No. PST.CLG.977311, and by the OTKA T037212, T037991. F. Kun acknowledges financial support of the Bólyai János and Békési György Foundations of the Hungarian Academy of Sciences and of the Research Contract No. FKFP 0118/2001.

APPENDIX

Here we provide the derivation of the average lifetime for the general case when the lifetime t_f of a bundle with a specific realization of the disorder can be cast in the form

$$t_f = \sum_{i=0}^{N-1} \left[G\left(\frac{i}{N}, x_{i+1}\right) - G\left(\frac{i}{N}, x_i\right) \right], \tag{A1}$$

i.e., t_f is a sum of terms that depend on the number of broken fibers i and on a single breaking threshold x_i that can be given as strain or stress. The x_i's are obtained by choosing N breaking thresholds independently from a cumulative probability distribution $P(x)$ and putting them into increasing order. This treatment includes both models discussed in the present paper. The expectation value of a function $f(x_i)$ can be determined as

$$\langle f(x_i)\rangle = \int \frac{N!}{(i-1)!(N-i)!} P(x)^{i-1}[1-P(x)]^{N-i} \times p(x) f(x) dx. \tag{A2}$$

The probability distribution in Eq. (A2) that the value of the ith largest breaking threshold falls between x and $x+dx$ has a sharp peak for large N values for each i. The above integration can be carried out by expanding the distribution about its peak. After expanding the result in terms of $1/N$ and neglecting higher-order terms, we arrive at

$$\langle f(x_i)\rangle = f(\overline{x_i}) + \frac{1}{2N} P(\overline{x_i})[1-P(\overline{x_i})] \times \left[\frac{f''(\overline{x_i})}{[P'(\overline{x_i})]^2} - \frac{f'(\overline{x_i})P''(\overline{x_i})}{[P'(\overline{x_i})]^3}\right], \tag{A3}$$

where $\overline{x_i}$ is defined implicitly by $P(\overline{x_i}) = i/(N+1)$. Applying Eq. (A3) to Eq. (A1) the resulting summation can be approximated by the integrals replacing i/N by the equivalent $P(\overline{x_i})(1+1/N)$. Neglecting corrections higher order in $1/N$ after straightforward calculations we arrive at

$$\langle t_f\rangle \approx \int dx \partial_2 G(P(x),x) + \frac{1}{2N}\int dx P(x)[1 - P(x)]\partial_1^2\partial_2 G(P(x),x), \tag{A4}$$

where $\partial_2 G(y,x) \equiv \partial G(y,x)/\partial x$, and $\partial_1^2\partial_2 G(y,x) \equiv \partial^3 G(y,x)/\partial^2 y \partial x$. Substituting the actual form of $G(y,x)$ for a specific model the complete form of the size scaling of lifetime can be obtained. However, it can be seen in the general expression, Eq. (A4), that the first term provides the lifetime of the infinite bundle and the only size dependence is in the prefactor of the second term. Equation (A4) states that if the lifetime can be written in the form of Eq. (A1) the lifetime of finite bundles converges to that of the infinite one as $1/N$ with increasing number of fibers N.

[1] T.L. Laufenberg, L.C. Palka, and J. Dobbin McNatt (unpublished).
[2] J.H. Pu, R.C. Tang, and W.C. Davis, Forest Products. J. **42**, 49 (1994).
[3] T.F. Drouillard and F.C. Beall, J. Acoust. Emiss. **9**, 215 (1990).
[4] C.C. Gerhards and C.L. Link, Wood Fiber Sci. **19**, 147 (1987).
[5] T.T. Chiao, C.C. Chiao, and R.J. Sherry, Frac. Mech. and Technol. **1**, 257 (1977).
[6] H. Otani, S.L. Phoenix, and P. Petrina, J. Mater. Sci. **26**, 1955 (1991).
[7] D.S. Farquhar, F.M. Mutselle, S.L. Phoenix, and R.L. Smith, J. Mater. Sci. **24**, 2151 (1989).
[8] M. Ibnabdeljalil and S.L. Phoenix, J. Mech. Phys. Solids **4**, 897 (1995).
[9] Z.Z. Du and R.M. McMeeking, J. Mech. Phys. Solids **43**, 701 (1995).
[10] B. Fabeny and W.A. Curtin, Acta Mater. **44**, 3439 (2001).
[11] B.D. Coleman, J. Appl. Phys. **29**, 968 (1958).
[12] H.E. Daniels, Proc. R. Soc. London, Ser. A **183**, 405 (1945).
[13] M. Kloster, A. Hansen, and P.C. Hemmer, Phys. Rev. E **56**, 2615 (1997).
[14] D.G. Harlow and S.L. Phoenix, J. Compos. Mater. **12**, 195 (1978).
[15] P.L. Leath and P.M. Duxbury, Phys. Rev. B **49**, 14 905 (1994).
[16] S.D. Zhang and E.J. Ding, Phys. Rev. B **53**, 646 (1996).
[17] A. Delaplace, S. Roux, and G. Pijaudier-Cabot, Int. J. Solids Struct. **36**, 1403 (1999).
[18] D. Sornette, J. Phys. A **22**, L243 (1989).
[19] D. Sornette, J. Phys.: Condens. Matter **50**, 745 (1989).
[20] J.V. Andersen, D. Sornette, and K.-T. Leung, Phys. Rev. Lett. **78**, 2140 (1997).
[21] F. Kun, S. Zapperi, and H.J. Herrmann, Eur. Phys. J. B **17**, 269 (2000).
[22] R.C. Hidalgo, F. Kun, and H.J. Herrmann, Phys. Rev. E **64**, 066122 (2001).
[23] Y. Moreno, J.B. Gomez, and A.F. Pacheco, Phys. Rev. Lett. **85**, 2865 (2000).
[24] L. Moral, Y. Moreno, J.B. Gómez, and A.F. Pacheco, Phys. Rev. E **63**, 066106 (2001).
[25] I.L. Menezes-Sobrinho, A.T. Bernardes, and J.G. Moreira, Phys. Rev. E **63**, 025104 (2001).
[26] F. Kun and H.J. Herrmann, J. Mater. Sci. **35**, 4685 (2000).
[27] R.C. Hidalgo, F. Kun, and H.J. Herrmann, Phys. Rev. E **65**, 032502 (2002).
[28] S. Pradhan and B.K. Chakrabarti, Phys. Rev. E **65**, 016113 (2001).
[29] S. Pradhan, P. Bhattacharyya, and B.K. Chakrabarti, Phys. Rev. E **66**, 016116 (2002).
[30] C.H. Weber, Z.Z. Du, and F.W. Zok, Acta Metall. Mater. **44**, 683 (1995).
[31] M. Tabichi, K. Kubo, A.T. Yokobori, and A. Fuji, Eng. Fract. Mech. **62**, 47 (1999).
[32] M.L. Gambone and A.H. Rosenberger, Acta Mater. **47**, 1723 (1999).
[33] L. Weber, P. Canalis-Nieto, A. Rossoll, and A. Mortensen, Acta Mater. **48**, 3235 (2000).
[34] A. Faucon *et al.*, Appl. Compos. Mater. **9**, 379 (2002).